Tree and Snake Polyominoes of Maximal Area

Alexandre Blondin Massé [1] *Alain Goupil* [2] *Mélodie Lapointe* [1]

[1]Laboratoire d'informatique formelle, Université du Québec à Chicoutimi, 555, boul. de l'Université, Chicoutimi, Québec, Canada, G7H 2B1

[2]Département de mathématiques et d'informatique, Université du Québec à Trois-Rivières

Abstract

Tree polyominoes, or lattice trees, are cycle-free polyominoes. In this article, we focus on the problem of generating tree polyominoes of maximal area $M(h, w)$ inscribed in a rectangle of height h and width w. By extending an algorithm of Jensen, we describe a method for generating such maximal trees based on lower and upper theoretical bounds for $M(h, w)$. We also study the families of tree polyominoes called kiss-free polyominoes and snake polyominoes i.e. trees with exactly two leaves.

Keywords: Polyomino, circumscribed rectangle, lattice tree, snake.

1 Introduction and Definitions

Polyominoes are geometrical objects that have recently been the subject of numerous investigations, either from a recreational [2], combinatorial or applied [3] perspective. They appear as models in different physical phenomenon such as percolation and the study of isomeres.

Simply stated, polyominoes are edge-connected sets of square cells of unit length in the discrete square plane considered up to translation. The area of a polyomino P is the number of cells it contains and we denote it by $c(P)$. In this paper we are interested in the study of *tree polyominoes*, also called *lattice trees* by some authors. They are characterized by the property that their dual edge graph is acyclic. *Snake polyominoes* form the subfamily of tree polyominoes having exactly two cells of degree one. They have been investigated in [4]. For sake of brevity, snake polyominoes and tree polyominoes are simply called *snakes* and *trees*. Similarly a *forest* is a collection of non edge-connected trees in the plane.

A tree polyomino is called *kiss-free* if it does not contain any 2×2 rectangle such that one diagonal is occupied and the other is not.

We consider polyominoes inside rectangular grid graphs which consist of $h \times w$ unit squares and $(h + 1) \times (w + 1)$ vertices (see Figure 1). A polyomino can be represented as a set of cells in the rectangular grid. Each cell in the rectangular grid is denoted by (i, j), where i is the row and j is the column as in the matrix index notation. The cell $(1, 1)$ is therefore in the upper left corner of the grid.

The *perimeter* of a polyomino P, denoted by $p(P)$, is the number of unit length edges on the boundary of P. The *point-perimeter* of P, denoted by $pp(P)$, is the number of vertices with integral coordinates lying on the boundary of P.

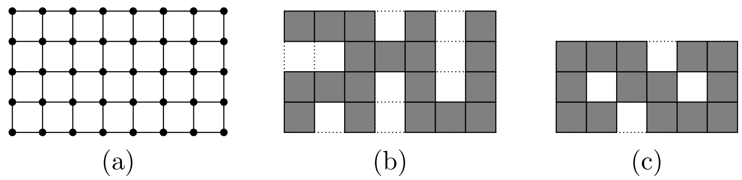

Figure 1: (a) The rectangular grid graph 5×7. (b) A kiss-free tree . (c) A snake with kissing number 2.

Notice that perimeter and point-perimeter coincide if and only if the polyomino is kiss-free. Otherwise the perimeter is larger than the point-perimeter. The *kissing number* $k(T)$ of a tree T is the number of corner only contacts between two cells c_i, c_j with at least two cells between them. For example the kissing number of the snake in Figure 1 c) is 2.

We end this introduction by proving elementary identities linking the area, the perimeter, the point perimeter and the kissing number of trees.

Proposition 1. *Let T be a nonempty tree. Then*

(i) $p(T) = 2(c(T) + 1)$;

(ii) $p(T) = pp(T) + k(T)$.

Proof. (i) By induction on $c(T)$. If $c(T) = 1$, then T has a single cell with perimeter 4 and the equality follows. Now, let T be a tree with $c(T)$ cells and let T' be a tree obtained by removing any leaf cell ℓ from T. Clearly, T' contains $c(T) - 1$ cells and its perimeter is $2c(T)$ by the induction hypothesis. Since ℓ is a leaf, 3 out of 4 of its edges contribute to the perimeter of T and one edge is attached to T so that the perimeter of T is $2c(T) + 3 - 1 = 2(c(T) + 1)$.

(ii) This follows directly from the definition of a kiss. $\square$

2 Kiss-Free Trees

Before addressing the problem of generating maximal trees and snakes inscribed in a given rectangle of dimensions $h \times w$ (Section 3), we concentrate on kiss-free trees and snakes. It turns out that the maximum area in that case is easy to establish, therefore solving a conjecture appearing in [4].

Theorem 1. *The maximal area of a kiss-free tree contained in a rectangle of size $h \times w$ is*

$$\left\lfloor \frac{(h+1)(w+1)}{2} \right\rfloor - 1. \tag{1}$$

Moreover the maximal area is realized by a snake for all values of h and w.

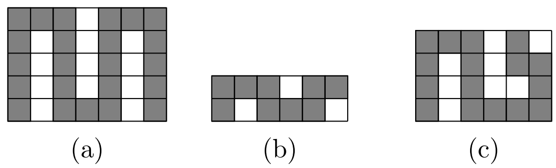

Figure 2: Kiss-free snakes of maximal area (a) 5×7 (b) 4×6 (c) 2×6.

Before proving Theorem 1, we construct, for any positive integers h and k, the following snakes. If w is odd (see Figure 2(a)), let

$$S(h, w) = \begin{cases} \{(i, w) \mid 1 \leq i \leq h\}, & \text{if } w = 1; \\ S(h, w-2) \cup (h, w-1) \cup \{(i, w) \mid 1 \leq i \leq h\}, & \text{if } w \equiv 1 \bmod 4; \\ S(h, w-2) \cup \{(1, w-1)\} \cup \{(i, w) \mid 1 \leq i \leq h\}, & \text{if } w \equiv 3 \bmod 4. \end{cases}$$

The snake $S(h, w)$, where h is odd, is defined symmetrically. Now, assume that w is even (see Figure 2(b)) and let

$$S(2, w) = \begin{cases} \{(1, 1), (2, 1), (1, 2)\}, & \text{if } w = 2; \\ S(2, w-2) \cup (1, w-1) \cup (2, w-1) \cup (1, w), & \text{if } w \equiv 0 \bmod 4; \\ S(2, w-2) \cup (1, w-1) \cup (2, w-1) \cup (2, w), & \text{if } w \equiv 2 \bmod 4. \end{cases}$$

Clearly, the snake $S(h, 2)$, where h is even is defined similarly. Finally if $h, w \geq 4$ are both even (see Figure 2(c)), let

$$S(h, w) = \begin{cases} S(h, w-3) \cup (1, w-2) \cup S(h, 2), & \text{if } w \equiv 0 \bmod 4; \\ S(h, w-3) \cup (h, w-2) \cup S(h, 2), & \text{if } w \equiv 2 \bmod 4. \end{cases}$$

Using the recursive definition and induction, it is straightforward to prove that the area of $S(h, w)$ is exactly $\lfloor (w+1)(h+1)/2 \rfloor - 1$. We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $c(T)$ be the area of a maximal kiss-free tree polyomino T inscribed in a $h \times w$ rectangle. The inequality $c(T) \geq \lfloor (w+1)(h+1)/2 \rfloor - 1$ follows from the construction of the snakes $S(h, w)$. It remains to prove that $c(T) \leq \lfloor (w+1)(h+1)/2 \rfloor - 1$.

We know from Proposition 1 that T has perimeter $2(c(T)+1)$. Moreover, since T is kiss-free, its point-perimeter is also $2(c(T)+1)$. But the rectangular grid graph of size $h \times w$ has exactly $(h+1)(w+1)$ vertices, therefore $2(c(T)+1) \leq (h+1)(w+1)$ which implies that

$$c(T) \leq \left\lfloor \frac{(h+1)(w+1)}{2} \right\rfloor - 1.$$

since $c(T)$ is an integer. $\square$

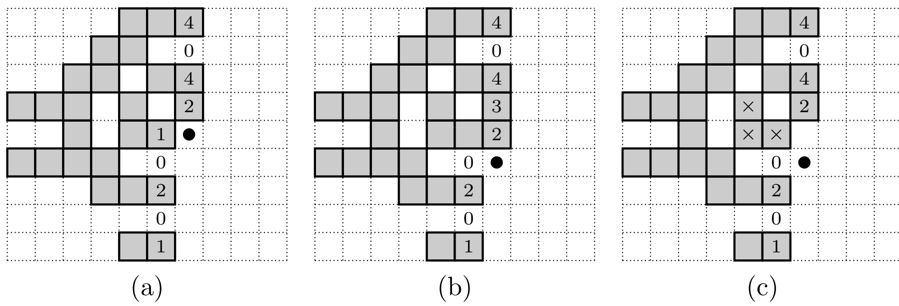

Figure 3: The basic idea of Jensen's algorithm. Black dots correspond to the current cell. (a) Some valid configuration. The status of each cell (0, 1, 2, 3 or 4) on the boundary is indicated. (b) An extension of the configuration in (a) by occupying the current cell. (c) An extension of the configuration in (a) obtained by leaving empty the current cell.

Theorem 1 is not that surprising when related to Hamiltonian cycles. Indeed, kiss-free trees of maximal area included in a $h \times w$ rectangle are in bijection with Hamiltonian cycles of the rectangular grid graph when at least one of the integers w, h is odd. However, if both integer dimensions of the rectangle are even, the grid graph has no Hamiltonian cycle but tree polyominoes of maximal area still exist. The problem of constructing and counting the number of Hamiltonian cycles in a rectangular grid graph has been investigated twenty years ago (see for instance [7]) and one-variable (w) ordinary generating functions for the first fixed values of h were expanded.

3 Trees and Snakes with Kisses

The problem of enumerating, counting or even computing the maximal area of trees and snakes inscribed in a given rectangle seems more involved. As a first step, for computational exploration purposes, it seems natural to design an algorithm for computing and enumerating such objects.

3.1 Computer Exploration

Different algorithms have been proposed for the enumeration of polyominoes [8, 1, 5, 6]. The most efficient known method is based on a transfer matrix, first introduced by Conway [1] and then improved by Jensen [5] and Knuth [6]. By adaptating Jensen's algorithm, it is not difficult to derive a method for enumerating snakes and trees of given area n inscribed in a given rectangle.

The basic idea of Jensen's algorithm is illustrated in Figure 3. It starts with an empty matrix and it fills its content with occupied/free cells columnwise, from top to bottom (see Figure 3). Information about connectedness can be compactly stored by considering only the boundary on the rightmost part of the partial configuration. The boundary is updated by a transfer matrix describing the rules for updating and discarding configurations. Both the current and above cell are updated to keep track of the connectedness. For instance, in Figure 3, configuration (b) is obtained from configuration (a) by occupying the current cell, and the upper and left cells are updated accordingly so that $2, 1$ becomes $3, 2$. By contrast, still in Figure 3, configuration (c) is obtained from (a) by leaving empty the current cell. The middle connected components marked with $\times$, can thus never be connected to the other component and the configuration is discarded.

When a cycle is created, it can be easily detected. In the same spirit, snakes can be enumerated by forbidding trees having more than two leaves. Finally, kisses can be detected locally by inspecting the current cell and its neighborhood.

A major advantage of Jensen's algorithm is that it prunes the search space very efficiently. More precisely, at each step, it computes the minimum number of cells that are needed in order to connect all parts of the current configuration, as well as making sure it touches the bottom, the top and the right side of the rectangle. If the number of cells needed is too high for the wished area, the configuration is then discarded. In other words, a *lower bound* is computed to discard unpromising configurations.

In the case where one wishes to enumerate *maximal* inscribed trees and snakes, the situation is the opposite, i.e. we need to compute an *upper bound* for the current configuration. More precisely, we determine a maximum number of cells that can be added, and if this number is smaller than the current best solution, then the configuration can be discarded. Such an upper bound is provided by Proposition 2 in Subsection 3.2.

3.2 Upper Bound of Trees Area

We now provide upper bounds for the value $M(h, w)$. As a first step, we show that at most $3/4$ of the rectangle area can be occupied, with a correction when either w or h is odd.

Proposition 2. *Let F be a forest inscribed in a $h \times w$ rectangle, with $h, w \geq 2$. Then*

$$c(F) \leq \frac{3(hw + h \bmod 2 + w \bmod 2)}{4}. \tag{2}$$

Proof. The different cases are shown in Figure 4.

If both h and w are even, then the rectangle can be partitioned in $hw/4$ subrectangles of dimensions 2×2. Since F does not contain any cycle, only 3 out of 4 cells may belong to the forest in each subrectangle. Therefore, $c(F) \leq 3hw/4$.

Now assume that one dimension is 3. With no loss of generality, let $w = 3$ and $h \geq 2$ be any integer. We prove by induction on h that Inequation (2) holds. Inspection shows that Inequation (2) holds for $h = 2, 3, 4, 5$. Now, let $h \geq 6$. If we partition the $h \times 3$ rectangle into two subrectangles of dimensions $(h-4) \times 3$ and 4×3, by induction hypothesis, we have that the area of F is

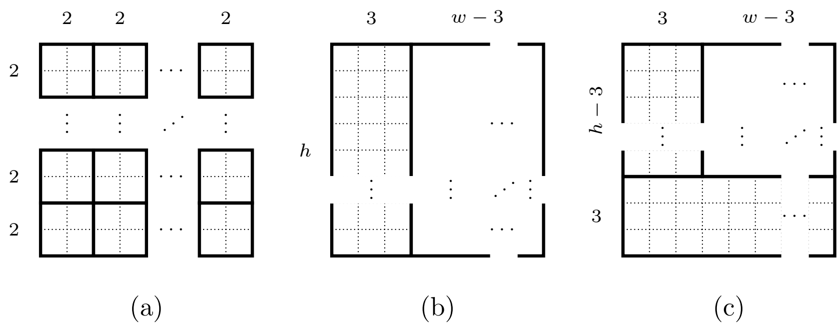

Figure 4: The three cases of Proposition 2. (a) If h and w are both even, then the rectangle can be divided in subrectangles of dimension 2×2. (b) If one dimension is even and the other is odd, then the rectangle can be divided in a subrectangle $h \times 3$ (or $3 \times w$) and a subrectangle $h \times (w-3)$ of even dimensions. (c) If both h and w are odd, then the rectangle is divided in subrectangles of dimensions $3 \times w$, $(h-3) \times 3$ and $(h-3) \times (w-3)$.

bounded by

$$\frac{3}{4}\left[(h-4) \cdot 3 + 1\right] + 9 = \frac{3}{4}(h \cdot 3 + 1),$$

so that Inequation (2) holds.

Consider the case where w is odd and h is even (the case w even, h odd is symmetric). Since the rectangle can be partitionned into two subrectangles of dimensions $h \times 3$ and $h \times (w-3)$ and by the two previous paragraphs, the area of F is bounded by

$$\frac{3}{4}\left[h \cdot 3 + 1\right] + \frac{3}{4}\left[h(w-3)\right] = \frac{3}{4}(hw + 1),$$

again verifying Inequation 2.

The case where both h and w are odd is follows from the previous paragraphs and the decomposition depicted in Figure 4(c). $\square$

Using the upper bound of Proposition 2 in our algorithm, we were able to compute different statistics found in Appendix A.

3.3 Lower Bound of Trees Maximal Area

Proposition 3. *Let $M(w,h)$ be the maximal area of a tree inscribed in a $h \times w$ rectangle. Then*

$$M(w,h) \geq \begin{cases} \left\lfloor \frac{2wh+w+h-1}{3} \right\rfloor - 3, & \text{if } w \text{ or } h \equiv 1 \bmod 3; \\ \left\lfloor \frac{2wh+w+h-1}{3} \right\rfloor - 2 - \left\lfloor \frac{(\min\{w,h\}-4)}{12} \right\rfloor, & \text{if } w \text{ or } h \equiv 0 \bmod 3; \\ \left\lfloor \frac{2wh+w+h-1}{3} \right\rfloor - 3 - \left\lfloor \frac{\min\{w,h\}-4}{6} \right\rfloor & \text{if } w \text{ and } h \equiv 2 \bmod 3. \end{cases}$$

Sketch of proof. The proof is constructive. We build a tree T with the given area in three steps. The first step is the construction of a polyomino in a strip

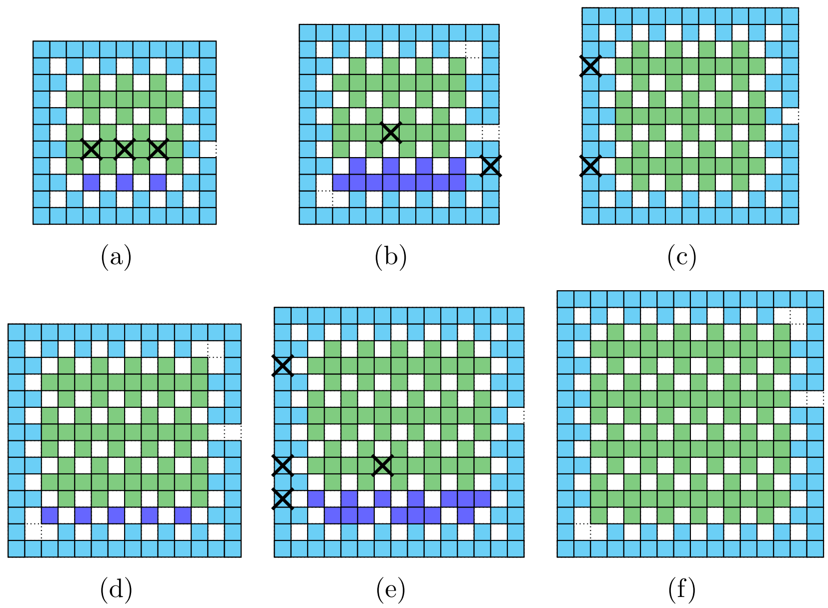

Figure 5: An example of forest polyominoes of almost maximal area for each case considered in the proof of Proposition 3. Region 1 appears in cyan, Region 2 in green and Region 3 in blue. All forests can be transformed into trees by moving each cell marked with an X.

of width two inside the rectangle and along the four sides of the rectangle. Call this rectangular strip Region 1 (colored in cyan in Figure 5). This polyomino is made of a first full rectangular strip of width one along the perimeter minus one cell to prevent the formation of a cycle. The second strip is also rectangular of width one inside the first strip and contains one cell of T in every other position of the strip. An easy calculation shows that the number of cells occupied by T in Region 1 is at least 3/4 of the area of Region 1 minus one depending on the parity of w and h. More precisely we have

$$c(T \cap \text{Region 1}) \geq \left\lfloor \frac{3}{4} c(\text{ Region 1}) \right\rfloor - 1$$

In the remaining rectangle of size $(h-2) \times (w-2)$, starting from the top, we form as many horizontal strips of heigth 3 as possible. This part of the rectangle is called Region 2 (colored in green in Figure 5) and the remaining part, called Region 3 (appearing in blue in Figure 5), is a rectangle of size $1 \times (w-4)$ or $2 \times (w-4)$. For each rectangle $3 \times (w-4)$ in Region 2, we exhibit a tree of area at least 2/3 the area of the rectangle minus one. If Region 3 is a $1 \times (w-4)$ rectangle, we can insert one cell in every other position of the row so that the set of cells covers 1/2 of the area of the rectangle. If Region 3 is a $2 \times (w-4)$ rectangle (see Figure 5(b)), we can insert a set of cells in the rectangle which covers 5/8 of the area of the rectangle. This is because the number of cells in each column follows a periodic pattern with period of the form $2, 1, 1, 1$. Adding all the proportions from each region, we obtain the expression in Proposition 3. At the end of this construction, the rectangle is filled with a forest of trees that can relatively be transformed into a tree by moving few cells. □

We conclude this discussion on a lower bound of trees of maximal area by observing that experimentation shows that the more compact formula

$$\left\lfloor \frac{(2h+1)(2w+1)-4}{6} \right\rfloor$$

is a tight approximation for the maximal area of inscribed trees and that in fact, it seems to differs from the real maximum by a small constant.

There is another observation that was made in the course of our experimentations and investigation. Indeed, it seems that a forest inscribed in a given rectangle can never have an area greater than the area of a tree of maximum area in the same rectangle:

Conjecture 1. *The area of a forest included in a $h \times w$ rectangle never exceeds the area of a single tree of maximal area inscribed in the same rectangle.*

Assuming Conjecture 1 is true, the speed of our enumeration algorithm could be significantly improved, since the values computed for smaller rectangles could be used as upper bounds when processing larger rectangles.

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A Computations

h \ w	2	3	4	5	6	7	8	9	10
2	3	5	6	8	9	11	12	14	15
3	5	7	9	11	14	16	18	20	22
4	6	9	11	14	17	20	22	25	28
5	8	11	14	17	21	24	27	30	34
6	9	14	17	21	24	29	32	36	40
7	11	16	20	24	29	33	38	42	46
8	12	18	22	27	32	38	42	48	52
9	14	20	25	30	36	42	48	?	?
10	15	22	28	34	40	46	52	?	?

Table 1: Maximal area of snakes inscribed in a rectangle $h \times w$.

h \ w	2	3	4	5	6	7	8	9	10
2	4	2	6	2	8	2	10	2	12
3	2	8	14	18	2	4	6	8	10
4	6	14	84	26	32	16	152	48	24
5	2	18	26	56	4	24	32	108	2
6	8	2	32	4	136	10	168	32	8
7	2	4	16	24	10	52	4	8	200
8	10	6	152	32	168	4	216	8	192
9	2	8	48	108	32	8	8	?	?
10	12	10	24	2	8	200	192	?	?

Table 2: Number of maximal snakes inscribed in a rectangle $h \times w$.

h \ w	2	3	4	5	6	7	8	9	10
2	3	5	6	8	9	11	12	14	15
3	5	7	9	12	14	16	18	21	23
4	6	9	12	15	18	21	24	27	30
5	8	12	15	19	22	26	30	34	37
6	9	14	18	22	26	31	35	39	44
7	11	16	21	26	31	36	41	46	51
8	12	18	24	30	35	41	46	52	58
9	14	21	27	34	39	46	52	59	65
10	15	23	30	37	44	51	58	65	72

Table 3: Maximal area of maximal trees inscribed in a rectangle $h \times w$.

h \ w	2	3	4	5	6	7	8	9	10
2	4	2	10	4	24	8	56	16	128
3	2	10	26	2	10	50	194	4	32
4	10	26	32	50	56	64	72	80	88
5	4	2	50	22	608	182	16	2	188
6	24	10	56	608	4120	208	1968	22716	168
7	8	50	64	182	208	488	560	1050	1096
8	56	194	72	16	1968	560	65864	14340	536
9	16	4	80	2	22716	1050	14340	166	3296
10	128	32	88	188	168	1096	536	3296	1296

Table 4: Number of maximal trees inscribed in a rectangle $h \times w$.

	2	3	4	5	6	7	8	9	10
2	3	5	6	8	9	11	12	14	15
3	5	7	9	11	13	15	17	19	21
4	6	9	11	14	16	19	21	24	26
5	8	11	14	17	20	23	26	29	32
6	9	13	16	20	23	27	30	34	37
7	11	15	19	23	27	31	35	39	43
8	12	17	21	26	30	35	39	44	?
9	14	19	24	29	34	?	?	?	?
10	15	21	26	32	?	?	?	?	?

Table 5: Maximal area of maximal kiss-free snakes inscribed in a rectangle $h \times w$.

	2	3	4	5	6	7	8	9	10
2	4	2	6	2	8	2	10	2	12
3	2	4	6	10	14	20	26	34	42
4	6	6	32	10	64	30	228	40	344
5	2	10	10	24	38	96	116	224	370
6	8	14	64	38	256	170	1096	416	3760
7	2	20	30	96	170	576	1098	2698	5322
8	10	26	228	116	1096	1098	14552	3628	?
9	2	34	40	224	416	?	?	?	?
10	12	42	344	370	?	?	?	?	?

Table 6: Number of maximal kiss-free snakes inscribed in a rectangle $h \times w$.

h \ w	2	3	4	5	6	7	8	9	10
2	3	5	6	8	9	11	12	14	15
3	5	7	9	11	13	15	17	19	21
4	6	9	11	14	16	19	21	24	26
5	8	11	14	17	20	23	26	29	32
6	9	13	16	20	23	27	30	34	37
7	11	15	19	23	27	31	35	39	?
8	12	17	21	26	30	35	39	?	?
9	14	19	24	29	34	?	?	?	?
10	15	21	26	32	37	?	?	?	?

Table 7: Maximal area maximal kiss-free trees inscribed in a rectangle $h \times w$.

h \ w	2	3	4	5	6	7	8	9	10
2	4	2	10	4	24	8	56	16	128
3	2	6	14	37	92	236	596	1517	3846
4	10	14	144	154	1984	1696	26252	18684	337640
5	4	37	154	1072	5320	32675	175294	1024028	5668692
6	24	92	1984	5320	145984	301384	10155528	17066492	681908296
7	8	236	1696	32675	301384	4638576	445348242	681728204	?
8	56	596	26252	175294	1015528	494483138	?	?	?
9	16	1517	18684	1024028	17066492	?	?	?	?
10	128	3846	337640	5668692	681908296	?	?	?	?

Table 8: Number of maximal kiss-free trees inscribed in a rectangle $h \times w$.