Enumerating minimum feedback vertex sets in directed graphs

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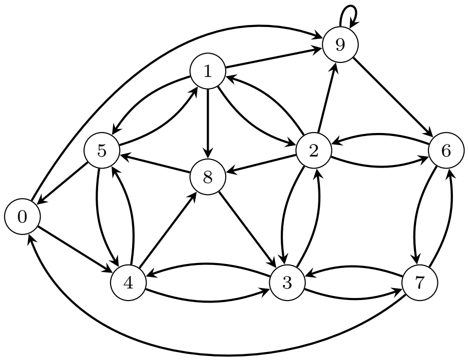
Extended Abstract

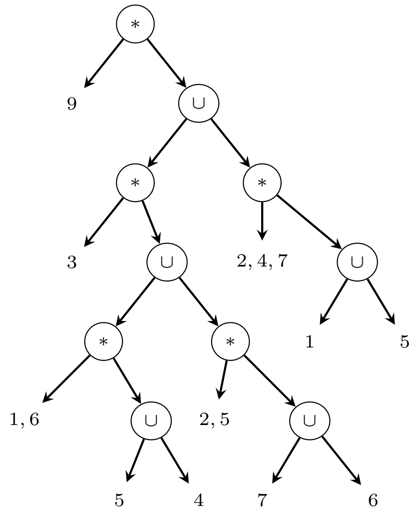
The minimum feedback vertex set (MFVS) problem for directed graph is one of those considered by Karp in his famous paper of 1972 [4]. Given a directed graph $G = (V, E)$, recall that a *feedback vertex set* is a subset S of V such that each circuit of G is covered by S. In other words, S is a subset of vertices such that the subgraph G' of G induced by $V - S$ is acyclic. The MFVS problem is known to be NP-hard for general classes of graphs [4], even in its versions for undirected graphs or for sets of arcs instead of vertices. It has been studied extensively and has numerous applications in electronics, computers-aided design, deadlock prevention, programming verification and Bayesian inference [1]. Recently, we have shown that it is a graph-theorical formulation of a fundamental linguistic problem called *symbol grounding problem* [2], which consists in finding minimum sets of words in a dictionary such that we may learn all remaining words starting from these initial sets. It is well-known that the number of minimum feedback vertex sets may be exponential in the size of the graph [3]. Therefore, one might wonder whether efficient data structures exist for storing all solutions in the case of particular classes of graphs.

In [1, 3], the authors consider the problem of finding one MFVS. For example, Lin and Jou introduced the concept of *contractible graphs*, that reduce to the empty graph under the application of eight contraction operators [1]. In particular, one can compute a MFVS in polynomial time whenever the initial graph is contractible and otherwise, they suggest a branch-and-bound algorithm based on their reduction operators. On the other hand, Formin et al. propose another algorithm to compute a MFVS for undirected graphs in $O(1.7548^n)$ by transforming the problem into that of finding maximum induced trees [3]. Finally, in [5], Formin et al. provide an algorithm for enumerating all minimal (inclusion-wise) feedback vertex sets with polynomial delay. To our knowledge, this is the only known non trivial algorithm related to the enumeration of MFVS's.

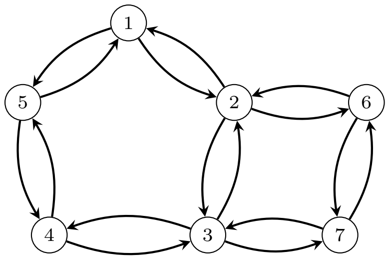
In this short communication, we propose an algorithm that enumerates all MFVS's (cardinality-wise), as well as a convenient data structure for representing them. As a first step, we restrict our attention on contractible graphs. Our algorithm is based on a branch-and-bound approach as well as the reduction operators of Lin and Jou [1]. Indeed, it turns out that 7 of 8 operators may be naturally extended to keep track of all solutions. More precisely, let $G = (V, E)$ be a directed graph (see Figure 1(a)). Given $v \in V$, we say that v is *essential* if it belongs to every MFVS of G. Similarly, v is called *useless* if it does not belong to any MFVS. At each step of the algorithm, we efficiently compute essential and useless vertices. On one hand, we remove essential vertices and add them to the current solution. On the other hand, useless vertices are removed while connecting each predecessor with each successor. Next we apply two operators introduced in [1] that act on edges only. When the graph is reduced as much as possible (see Figure 1(b)), we can branch according to any remaining vertex, by including or excluding it from the current partial MFVS. Also, whenever it is possible, we break the problem into smaller parts according to the connected components of the graph, which speeds up significantly the computations.



(a) A directed graph



(c) A tree representing all six solutions



(b) The directed graph obtained after reduction

Figure 1: An example. In (c), '$\cup$' represents the union operator and '$*$' represents the concatenation operator, while the leaves are sets of vertices.

Since the number of solutions is very large, it is convenient to use a regular expression to represent all the solutions. For instance, a regular expression describing every MFVS of the graph of Figure 1(a) is represented as a tree in Figure 1(c). This representation presents many advantages such as generating efficiently random MFVS's with uniform probability, iterating over all solutions with constant delay, compute the number of solutions in which each vertex appears and verify whether a given set is a MFVS or not.

References

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